CHAPTER 12

Case Studies: Linear Algebraic Equations

The purpose of this chapter is to use the numerical procedures discussed in Chaps. 9, 10, and 11 to solve systems of linear algebraic equations for some engineering case studies. These systematic numerical techniques have practical significance because engineers frequently encounter problems involving systems of equations that are too large to solve by hand. The numerical algorithms in these applications are particularly convenient to implement on personal computers.

Section 12.1 shows how a mass balance can be employed to model a system of reactors. *Section 12.2* places special emphasis on the use of the matrix inverse to determine the complex cause-effect interactions between forces in the members of a truss. *Section 12.3* is an example of the use of Kirchhoff's laws to compute the currents and voltages in a resistor circuit. Finally, *Sec. 12.4* is an illustration of how linear equations are employed to determine the steady-state configuration of a mass-spring system.

12.1 STEADY-STATE ANALYSIS OF A SYSTEM OF REACTORS (CHEMICAL/BIO ENGINEERING)

Background. One of the most important organizing principles in chemical engineering is the *conservation of mass* (recall Table 1.1). In quantitative terms, the principle is expressed as a mass balance that accounts for all sources and sinks of a material that pass in and out of a volume (Fig. 12.1). Over a finite period of time, this can be expressed as

$$\text{Accumulation} = \text{inputs} - \text{outputs} \tag{12.1}$$

The mass balance represents a bookkeeping exercise for the particular substance being modeled. For the period of the computation, if the inputs are greater than the outputs, the mass of the substance within the volume increases. If the outputs are greater than the inputs, the mass decreases. If inputs are equal to the outputs, accumulation is zero and mass remains constant. For this stable condition, or steady state, Eq. (12.1) can be expressed as

$$\text{Inputs} = \text{outputs} \tag{12.2}$$

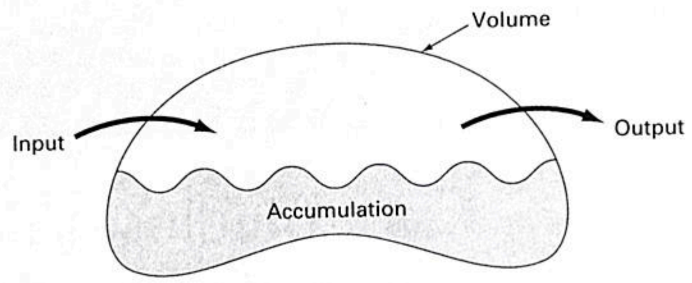


FIGURE 12.1
A schematic representation of mass balance.

Employ the conservation of mass to determine the steady-state concentrations of a system of coupled reactors.

Solution. The mass balance can be used for engineering problem solving by expressing the inputs and outputs in terms of measurable variables and parameters. For example, if we were performing a mass balance for a conservative substance (that is, one that does not increase or decrease due to chemical transformations) in a reactor (Fig. 12.2), we would have to quantify the rate at which mass flows into the reactor through the two inflow pipes and out of the reactor through the outflow pipe. This can be done by taking the product of the flow rate Q (in cubic meters per minute) and the concentration c (in milligrams per cubic meter) for each pipe. For example, for pipe 1 in Fig. 12.2, $Q_1 = 2$ m^3/min and $c_1 = 25$ mg/m^3; therefore the rate at which mass flows into the reactor through pipe 1 is $Q_1c_1 = (2\ \text{m}^3/\text{min})(25\ \text{mg/m}^3) = 50$ mg/min. Thus, 50 mg of chemical flows into the reactor through this pipe each minute. Similarly, for pipe 2 the mass inflow rate can be calculated as $Q_2c_2 = (1.5\ \text{m}^3/\text{min})(10\ \text{mg/m}^3) = 15$ mg/min.

Notice that the concentration out of the reactor through pipe 3 is not specified by Fig. 12.2. This is because we already have sufficient information to calculate it on the basis of the conservation of mass. Because the reactor is at steady state, Eq. (12.2) holds and the inputs should be in balance with the outputs, as in

$$Q_1c_1 + Q_2c_2 = Q_3c_3$$

Substituting the given values into this equation yields

$$50 + 15 = 3.5c_3$$

which can be solved for $c_3 = 18.6$ mg/m^3. Thus, we have determined the concentration in the third pipe. However, the computation yields an additional bonus. Because the reactor is well mixed (as represented by the propeller in Fig. 12.2), the concentration will be uniform, or homogeneous, throughout the tank. Therefore the concentration in pipe 3 should be identical to the concentration throughout the reactor. Consequently, the mass balance has allowed us to compute both the concentration in the reactor and in the outflow pipe. Such

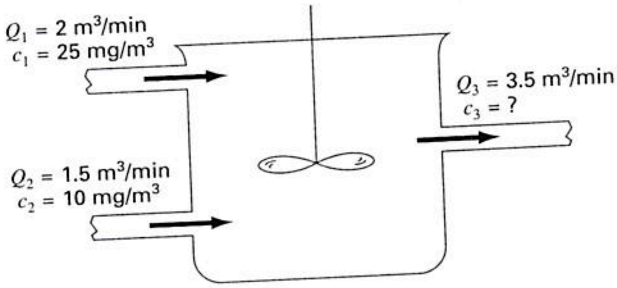


FIGURE 12.2
A steady-state, completely mixed reactor with two inflow pipes and one outflow pipe. The flows Q are in cubic meters per minute, and the concentrations c are in milligrams per cubic meter.

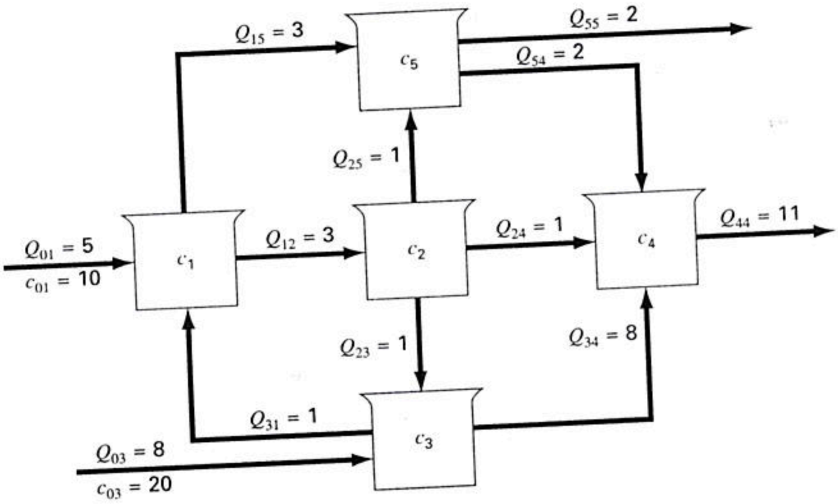


FIGURE 12.3
Five reactors linked by pipes.

information is of great utility to chemical and petroleum engineers who must design reactors to yield mixtures of a specified concentration.

Because simple algebra was used to determine the concentration for the single reactor in Fig. 12.2, it might not be obvious how computers figure in mass-balance calculations. Figure 12.3 shows a problem setting where computers are not only useful but are a practical necessity. Because there are five interconnected, or coupled, reactors, five simultaneous mass-balance equations are needed to characterize the system. For reactor 1, the rate of mass flow in is

$$5(10) + Q_{31}c_3$$

and the rate of mass flow out is

$$Q_{12}c_1 + Q_{15}c_1$$

Because the system is at steady state, the inflows and outflows must be equal:

$$5(10) + Q_{31}c_3 = Q_{12}c_1 + Q_{15}c_1$$

or, substituting the values for flow from Fig. 12.3,

$$6c_1 - c_3 = 50$$

Similar equations can be developed for the other reactors:

$$-3c_1 + 3c_2 = 0$$
$$-c_2 + 9c_3 = 160$$
$$-c_2 - 8c_3 + 11c_4 - 2c_5 = 0$$
$$-3c_1 - c_2 + 4c_5 = 0$$

A numerical method can be used to solve these five equations for the five unknow concentrations:

$$\{C\}^T = \lfloor 11.51 \quad 11.51 \quad 19.06 \quad 17.00 \quad 11.51 \rfloor$$

In addition, the matrix inverse can be computed as

$$[A]^{-1} = \begin{bmatrix} 0.16981 & 0.00629 & 0.01887 & 0 & 0 \\ 0.16981 & 0.33962 & 0.01887 & 0 & 0 \\ 0.01887 & 0.03774 & 0.11321 & 0 & 0 \\ 0.06003 & 0.07461 & 0.08748 & 0.09091 & 0.04545 \\ 0.16981 & 0.08962 & 0.01887 & 0 & 0.25000 \end{bmatrix}$$

Each of the elements a_{ij} signifies the change in concentration of reactor i due to a u change in loading to reactor j. Thus, the zeros in column 4 indicate that a loading to re tor 4 will have no impact on reactors 1, 2, 3, and 5. This is consistent with the system c figuration (Fig. 12.3), which indicates that flow out of reactor 4 does not feed back into of the other reactors. In contrast, loadings to any of the first three reactors will affect entire system as indicated by the lack of zeros in the first three columns. Such informa is of great utility to engineers who design and manage such systems.

12.2 ANALYSIS OF A STATICALLY DETERMINATE TRUSS (CIVIL/ENVIRONMENTAL ENGINEERING)

Background. An important problem in structural engineering is that of finding the f and reactions associated with a statically determinate truss. Figure 12.4 shows an exa of such a truss.

The forces (F) represent either tension or compression on the members of the trus ternal reactions (H_2, V_2, and V_3) are forces that characterize how the truss interacts wi supporting surface. The hinge at node 2 can transmit both horizontal and vertical forces surface, whereas the roller at node 3 transmits only vertical forces. It is observed that fect of the external loading of 1000 lb is distributed among the various members of the

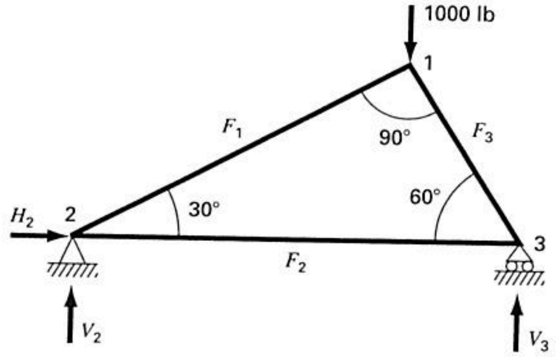
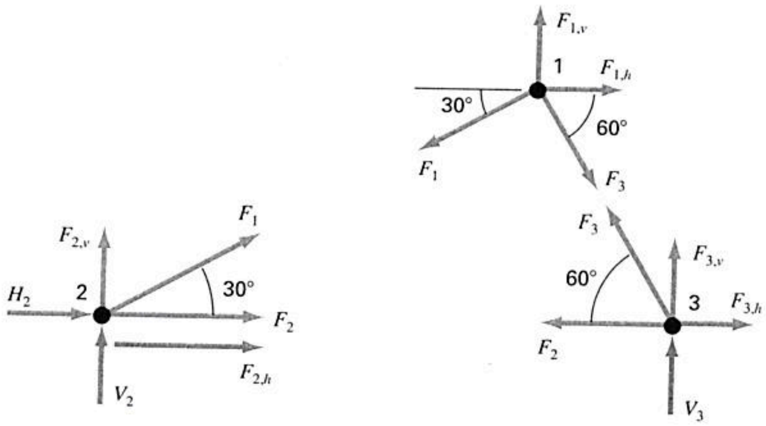


FIGURE 12.4
Forces on a statically determi-
nate truss.



URE 12.5
-body force diagrams for
odes of a statically
rminate truss.

Solution. This type of structure can be described as a system of coupled linear algebraic equations. Free-body force diagrams are shown for each node in Fig. 12.5. The sum of the forces in both horizontal and vertical directions must be zero at each node, because the system is at rest. Therefore, for node 1,

$$\Sigma F_H = 0 = -F_1 \cos 30^\circ + F_3 \cos 60^\circ + F_{1,h} \tag{12.3}$$
$$\Sigma F_V = 0 = -F_1 \sin 30^\circ - F_3 \sin 60^\circ + F_{1,v} \tag{12.4}$$

for node 2,

$$\Sigma F_H = 0 = F_2 + F_1 \cos 30^\circ + F_{2,h} + H_2 \tag{12.5}$$
$$\Sigma F_V = 0 = F_1 \sin 30^\circ + F_{2,v} + V_2 \tag{12.6}$$

for node 3,

$$\Sigma F_H = 0 = -F_2 - F_3 \cos 60^\circ + F_{3,h} \tag{12.7}$$

$$\Sigma F_V = 0 = F_3 \sin 60^\circ + F_{3,v} + V_3 \tag{12.8}$$

where $F_{i,h}$ is the external horizontal force applied to node i (where a positive force is from left to right) and $F_{1,v}$ is the external vertical force applied to node i (where a positive force is upward). Thus, in this problem, the 1000-lb downward force on node 1 corresponds to $F_{1,v} = -1000$. For this case all other $F_{i,v}$'s and $F_{i,h}$'s are zero. Note that the directions of the internal forces and reactions are unknown. Proper application of Newton's laws requires only consistent assumptions regarding direction. Solutions are negative if the directions are assumed incorrectly. Also note that in this problem, the forces in all members are assumed to be in tension and act to pull adjoining nodes together. A negative solution therefore corresponds to compression. This problem can be written as the following system of six equations and six unknowns:

$$\begin{bmatrix} 0.866 & 0 & -0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.866 & 0 & 0 & 0 \\ -0.866 & -1 & 0 & -1 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -0.866 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ H_2 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12.9}$$

Notice that, as formulated in Eq. (12.9), partial pivoting is required to avoid division by zero diagonal elements. Employing a pivot strategy, the system can be solved using any of the elimination techniques discussed in Chap. 9 or 10. However, because this problem is an ideal case study for demonstrating the utility of the matrix inverse, the *LU* decomposition can be used to compute

$$F_1 = -500 \qquad F_2 = 433 \qquad F_3 = -866$$
$$H_2 = 0 \qquad V_2 = 250 \qquad V_3 = 750$$

and the matrix inverse is

$$[A]^{-1} = \begin{bmatrix} 0.866 & 0.5 & 0 & 0 & 0 & 0 \\ 0.25 & -0.433 & 0 & 0 & 1 & 0 \\ -0.5 & 0.866 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ -0.433 & -0.25 & 0 & -1 & 0 & 0 \\ 0.433 & -0.75 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Now, realize that the right-hand-side vector represents the externally applied horizontal and vertical forces on each node, as in

$$\{F\}^T = \lfloor F_{1,h} \quad F_{1,v} \quad F_{2,h} \quad F_{2,v} \quad F_{3,h} \quad F_{3,v} \rfloor \tag{12.10}$$

Because the external forces have no effect on the *LU* decomposition, the method need not be implemented over and over again to study the effect of different external forces on the truss. Rather, all that we have to do is perform the forward- and backward-substitution steps for each right-hand-side vector to efficiently obtain alternative solutions. For example,

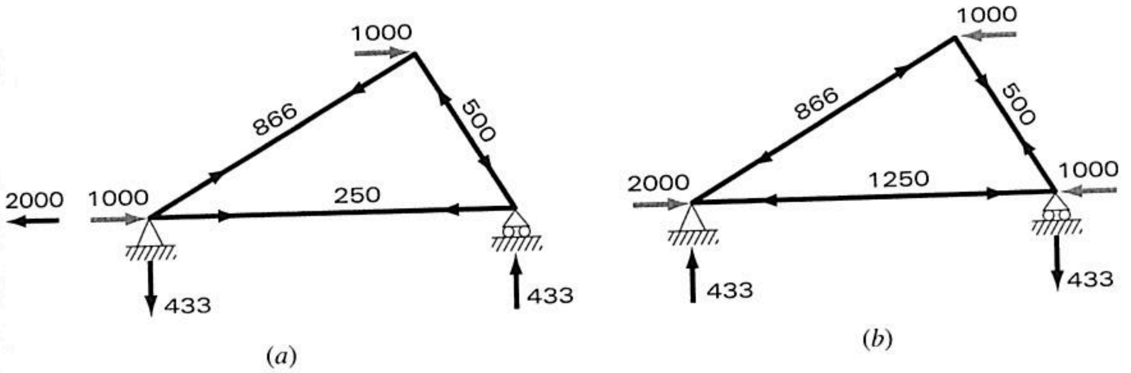


FIGURE 12.6
Two test cases showing (a) winds from the left and (b) winds from the right.

we might want to study the effect of horizontal forces induced by a wind blowing from left to right. If the wind force can be idealized as two point forces of 1000 lb on nodes 1 and 2 (Fig. 12.6*a*), the right-hand-side vector is

$$\{F\}^T = \lfloor -1000 \quad 0 \quad 1000 \quad 0 \quad 0 \quad 0 \rfloor$$

which can be used to compute

$$F_1 = 866 \qquad F_2 = 250 \qquad F_3 = -500$$
$$H_2 = -2000 \qquad V_2 = -433 \qquad V_3 = 433$$

For a wind from the right (Fig. 12.6*b*), $F_{1,h} = -1000$, $F_{3,h} = -1000$, and all other external forces are zero, with the result that

$$F_1 = -866 \qquad F_2 = -1250 \qquad F_3 = 500$$
$$H_2 = 2000 \qquad V_2 = 433 \qquad V_3 = -433$$

The results indicate that the winds have markedly different effects on the structure. Both cases are depicted in Fig. 12.6.

The individual elements of the inverted matrix also have direct utility in elucidating stimulus-response interactions for the structure. Each element represents the change of one of the unknown variables to a unit change of one of the external stimuli. For example, element a_{32}^{-1} indicates that the third unknown (F_3) will change 0.866 due to a unit change of the second external stimulus ($F_{1,v}$). Thus, if the vertical load at the first node were increased by 1, F_3 would increase by 0.866. The fact that elements are 0 indicates that certain unknowns are unaffected by some of the external stimuli. For instance $a_{13}^{-1} = 0$ means that F_1 is unaffected by changes in $F_{2,h}$. This ability to isolate interactions has a number of engineering applications, including the identification of those components that are most sensitive to external stimuli and, as a consequence, most prone to failure. In addition, it can be used to determine components that may be unnecessary (see Prob. 12.18).

The foregoing approach becomes particularly useful when applied to large comp structures. In engineering practice, it may be necessary to solve trusses with hundreds even thousands of structural members. Linear equations provide one powerful approach gaining insight into the behavior of these structures.

12.3 CURRENTS AND VOLTAGES IN RESISTOR CIRCUITS (ELECTRICAL ENGINEERING)

Background. A common problem in electrical engineering involves determining t currents and voltages at various locations in resistor circuits. These problems are solv using Kirchhoff's current and voltage rules. The current (or point) rule states that the alg braic sum of all currents entering a node must be zero (see Fig. 12.7*a*), or

$$\Sigma i = 0 \tag{12.11}$$

where all current entering the node is considered positive in sign. The current rule is a application of the principle of conservation of charge (recall Table 1.1).

The voltage (or loop) rule specifies that the algebraic sum of the potential difference (that is, voltage changes) in any loop must equal zero. For a resistor circuit, this i expressed as

$$\Sigma\xi - \Sigma i R = 0 \tag{12.12}$$

where ξ is the emf (electromotive force) of the voltage sources and R is the resistance of any resistors on the loop. Note that the second term derives from Ohm's law (Fig. 12.7*b*), which states that the voltage drop across an ideal resistor is equal to the product of the cur- rent and the resistance. Kirchhoff's voltage rule is an expression of the conservation of energy.

Solution. Application of these rules results in systems of simultaneous linear algebraic equations because the various loops within a circuit are coupled. For example, consider the circuit shown in Fig. 12.8. The currents associated with this circuit are unknown both in magnitude and direction. This presents no great difficulty because one simply assumes a direction for each current. If the resultant solution from Kirchhoff's laws is negative, then the assumed direction was incorrect. For example, Fig. 12.9 shows some assumed currents.

FIGURE 12.7
Schematic representations of (*a*) Kirchhoff's current rule and (*b*) Ohm's law.

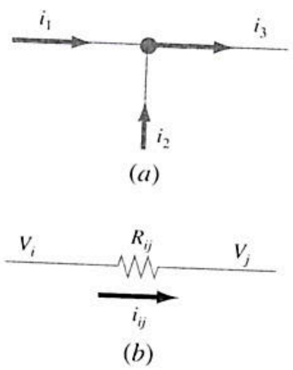
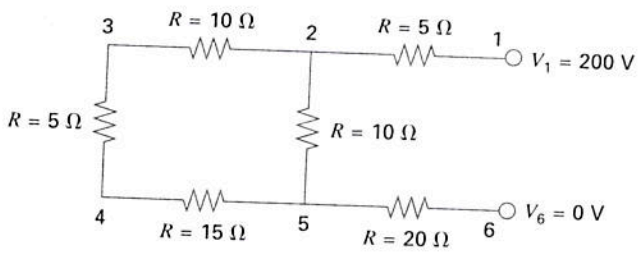


FIGURE 12.8
A resistor circuit to be solved using simultaneous linear algebraic equations.



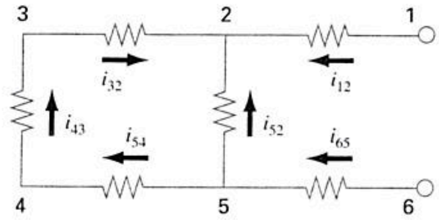


FIGURE 12.9
Assumed currents.

Given these assumptions, Kirchhoff's current rule is applied at each node to yield

$$i_{12} + i_{52} + i_{32} = 0$$
$$i_{65} - i_{52} - i_{54} = 0$$
$$i_{43} - i_{32} = 0$$
$$i_{54} - i_{43} = 0$$

Application of the voltage rule to each of the two loops gives

$$-i_{54}R_{54} - i_{43}R_{43} - i_{32}R_{32} + i_{52}R_{52} = 0$$
$$-i_{65}R_{65} - i_{52}R_{52} + i_{12}R_{12} - 200 = 0$$

or, substituting the resistances from Fig. 12.8 and bringing constants to the right-hand side,

$$-15i_{54} - 5i_{43} - 10i_{32} + 10i_{52} = 0$$
$$-20i_{65} - 10i_{52} + 5i_{12} = 200$$

Therefore, the problem amounts to solving the following set of six equations with six unknown currents:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 10 & -10 & 0 & -15 & -5 \\ 5 & -10 & 0 & -20 & 0 & 0 \end{bmatrix} \begin{Bmatrix} i_{12} \\ i_{52} \\ i_{32} \\ i_{65} \\ i_{54} \\ i_{43} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 200 \end{Bmatrix}$$

Although impractical to solve by hand, this system is easily handled using an elimination method. Proceeding in this manner, the solution is

$$i_{12} = 6.1538 \qquad i_{52} = -4.6154 \qquad i_{32} = -1.5385$$
$$i_{65} = -6.1538 \qquad i_{54} = -1.5385 \qquad i_{43} = -1.5385$$

Thus, with proper interpretation of the signs of the result, the circuit currents and voltages are as shown in Fig. 12.10. The advantages of using numerical algorithms and computers for problems of this type should be evident.

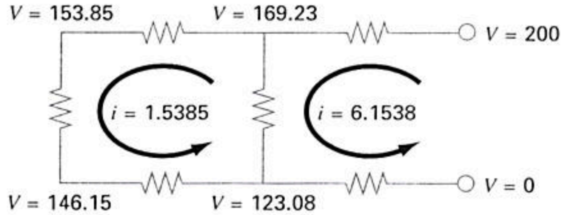


FIGURE 12.10
The solution for currents and voltages obtained using an elimination method.

12.4 SPRING-MASS SYSTEMS (MECHANICAL/AEROSPACE ENGINEERING)

Background. Idealized spring-mass systems play an important role in mechanical and other engineering problems. Figure 12.11 shows such a system. After they are released, the masses are pulled downward by the force of gravity. Notice that the resulting displacement of each spring in Fig. 12.11*b* is measured along local coordinates referenced to its initial position in Fig. 12.11*a*.

As introduced in Chap. 1, Newton's second law can be employed in conjunction with force balances to develop a mathematical model of the system. For each mass, the second law can be expressed as

$$m\frac{d^2x}{dt^2} = F_D - F_U \tag{12.13}$$

To simplify the analysis, we will assume that all the springs are identical and follow Hooke's law. A free-body diagram for the first mass is depicted in Fig. 12.12*a*. The upward force is merely a direct expression of Hooke's law:

$$F_U = kx_1 \tag{12.14}$$

The downward component consists of the two spring forces along with the action of gravity on the mass,

$$F_D = k(x_2 - x_1) + k(x_2 - x_1) = m_1 g \tag{12.15}$$

Note how the force component of the two springs is proportional to the displacement of the second mass, x_2, corrected for the displacement of the first mass, x_1.

Equations (12.14) and (12.15) can be substituted into Eq. (12.13) to give

$$m_1\frac{d^2x_1}{dt^2} = 2k(x_2 - x_1) + m_1 g - kx_1 \tag{12.16}$$

Thus, we have derived a second-order ordinary differential equation to describe the displacement of the first mass with respect to time. However, notice that the solution cannot be obtained because the model includes a second dependent variable, x_2. Consequently, free-body diagrams must be developed for the second and the third masses (Fig. 12.12*b* and *c*)

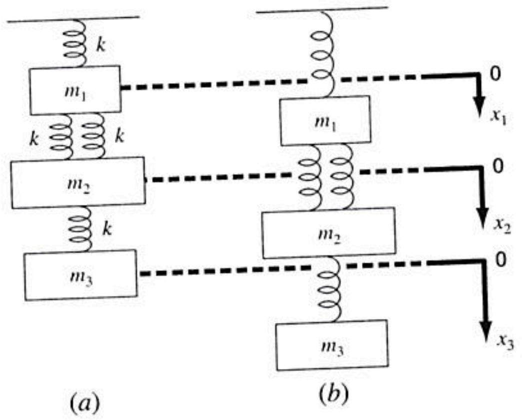


FIGURE 12.11
A system composed of three masses suspended vertically by a series of springs. (*a*) The system before release, that is, prior to extension or compression of the springs. (*b*) The system after release. Note that the positions of the masses are referenced to local coordinates with origins at their position before release.

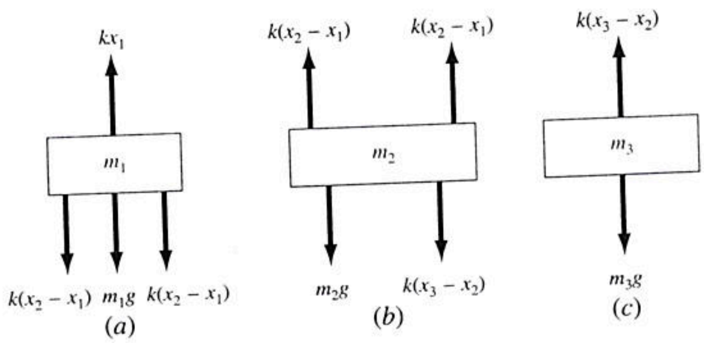


FIGURE 12.12
Free-body diagrams for the three masses from Fig. 12.11.

that can be employed to derive

$$m_2 \frac{d^2x_2}{dt^2} = k(x_3 - x_2) + m_2 g - 2k(x_2 - x_1) \tag{12.17}$$

and

$$m_3 \frac{d^2x_3}{dt^2} = m_3 g - k(x_3 - x_2) \tag{12.18}$$

Equations (12.16), (12.17), and (12.18) form a system of three differential equations with three unknowns. With the appropriate initial conditions, they could be used to solve for the displacements of the masses as a function of time (that is, their oscillations). We will discuss numerical methods for obtaining such solutions in Part Seven. For the present, we can obtain the displacements that occur when the system eventually comes to rest, that is, to the steady state. To do this, the derivatives in Eqs. (12.16), (12.17), and (12.18) are set to zero to give

$$\begin{aligned} 3kx_1 - 2kx_2 \qquad\qquad &= m_1 g \\ -2kx_1 + 3kx_2 - kx_3 &= m_2 g \\ - kx_2 + kx_3 &= m_3 g \end{aligned}$$

or, in matrix form,

$$[K]\{X\} = \{W\}$$

where $[K]$, called the *stiffness matrix*, is

$$[K] = \begin{bmatrix} 3k & -2k & \\ -2k & 3k & -k \\ & -k & k \end{bmatrix}$$

and $\{X\}$ and $\{W\}$ are the column vectors of the unknowns X and the weights mg, respectively.

Solution. At this point, numerical methods can be employed to obtain a solution. If $m_1 = 2$ kg, $m_2 = 3$ kg, $m_3 = 2.5$ kg, and the k's $= 10$ kg/s^2, use LU decomposition to solve for the displacements and generate the inverse of $[K]$.

Substituting the model parameters gives

$$[K] = \begin{bmatrix} 30 & -20 & \\ -20 & 30 & -10 \\ & -10 & 10 \end{bmatrix} \qquad \{W\} = \begin{Bmatrix} 19.6 \\ 29.4 \\ 24.5 \end{Bmatrix}$$

LU decomposition can be employed to solve for $x_1 = 7.35$, $x_2 = 10.045$, and $x_3 = 12.495$. These displacements were used to construct Fig. 12.11b. The inverse of the stiffness matrix is computed as

$$[K]^{-1} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.15 & 0.15 \\ 0.1 & 0.15 & 0.25 \end{bmatrix}$$

Each element of this matrix k_{ji}^{-1} tells us the displacement of mass i due to a unit force imposed on mass j. Thus, the values of 0.1 in column 1 tell us that a downward unit load to the first mass will displace all of the masses 0.1 m downward. The other elements can be interpreted in a similar fashion. Therefore, the inverse of the stiffness matrix provides a fundamental summary of how the system's components respond to externally applied forces.