boundary-value problems using finite differences and steady-state solutions of the one-dimensional problem with the finite-element approach (Chap. 31).

27.2 EIGENVALUE PROBLEMS

Eigenvalue, or characteristic-value, problems are a special class of boundary-value problems that are common in engineering problem contexts involving vibrations, elasticity, and other oscillating systems. In addition, they are used in a wide variety of engineering contexts beyond boundary-value problems. Before describing numerical methods for solving these problems, we will present some general background information. This includes discussion of both the mathematics and the engineering significance of eigenvalues.

27.2.1 Mathematical Background

Part Three dealt with methods for solving sets of linear algebraic equations of the general form

$$[A]\{X\} = \{B\}$$

Such systems are called *nonhomogeneous* because of the presence of the vector $\{B\}$ on the right-hand side of the equality. If the equations comprising such a system are linearly independent (that is, have a nonzero determinant), they will have a unique solution. In other words, there is one set of x values that will make the equations balance.

In contrast, a homogeneous linear algebraic system has the general form

$$[A]\{X\} = 0$$

Although nontrivial solutions (that is, solutions other than all x's $= 0$) of such systems are possible, they are generally not unique. Rather, the simultaneous equations establish relationships among the x's that can be satisfied by various combinations of values.

Eigenvalue problems associated with engineering are typically of the general form

$$\begin{aligned}
(a_{11} - \lambda)x_1 + \quad & a_{12}x_2 + \cdots + \quad & a_{1n}x_n = 0 \\
a_{21}x_1 + \ & (a_{22} - \lambda)x_2 + \cdots + \quad & a_{2n}x_n = 0 \\
\cdot \quad & \cdot \quad & \cdot \quad \cdot \\
\cdot \quad & \cdot \quad & \cdot \quad \cdot \\
\cdot \quad & \cdot \quad & \cdot \quad \cdot \\
a_{n1}x_1 + \quad & a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n = 0
\end{aligned}$$

where λ is an unknown parameter called the *eigenvalue*, or *characteristic value*. A solution $\{X\}$ for such a system is referred to as an *eigenvector*. The above set of equations may also be expressed concisely as

$$\big[[A] - \lambda[I]\big]\{X\} = 0 \tag{27.4}$$

The solution of Eq. (27.4) hinges on determining λ. One way to accomplish this is based on the fact that the determinant of the matrix $[[A] - \lambda[I]]$ must equal zero for nontrivial solutions to be possible. Expanding the determinant yields a polynomial in λ. The roots of this polynomial are the solutions for the eigenvalues. An example of this approach will be provided in the next section.

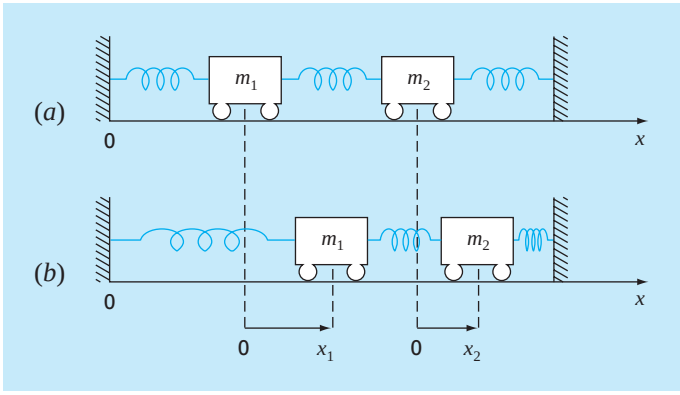

FIGURE 27.5
Positioning the masses away from equilibrium creates forces in the springs that upon release lead to oscillations of the masses. The positions of the masses can be referenced to local coordinates with origins at their respective equilibrium positions.

27.2.2 Physical Background

The mass-spring system in Fig. 27.5*a* is a simple context to illustrate how eigenvalues occur in physical problem settings. It also will help to illustrate some of the mathematical concepts introduced in the previous section.

To simplify the analysis, assume that each mass has no external or damping forces acting on it. In addition, assume that each spring has the same natural length l and the same spring constant k. Finally, assume that the displacement of each spring is measured relative to its own local coordinate system with an origin at the spring's equilibrium position (Fig. 27.5*a*). Under these assumptions, Newton's second law can be employed to develop a force balance for each mass (recall Sec. 12.4),

$$m_1 \frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$

and

$$m_2 \frac{d^2x_2}{dt^2} = -k(x_2 - x_1) - kx_2$$

where x_i is the displacement of mass i away from its equilibrium position (Fig. 27.5*b*). These equations can be expressed as

$$m_1 \frac{d^2x_1}{dt^2} - k(-2x_1 + x_2) = 0 \tag{27.5a}$$

$$m_2 \frac{d^2x_2}{dt^2} - k(x_1 - 2x_2) = 0 \tag{27.5b}$$

From vibration theory, it is known that solutions to Eq. (27.5) can take the form

$$x_i = A_i \sin(\omega t) \tag{27.6}$$

where A_i = the amplitude of the vibration of mass i and ω = the frequency of the vibration, which is equal to

$$\omega = \frac{2\pi}{T_p} \tag{27.7}$$

where T_p is the period. From Eq. (27.6) it follows that

$$x_i'' = -A_i \omega^2 \sin(\omega t) \tag{27.8}$$

Equations (27.6) and (27.8) can be substituted into Eq. (27.5), which, after collection of terms, can be expressed as

$$\left(\frac{2k}{m_1} - \omega^2\right) A_1 - \frac{k}{m_1} A_2 = 0 \tag{27.9a}$$

$$-\frac{k}{m_2} A_1 + \left(\frac{2k}{m_2} - \omega^2\right) A_2 = 0 \tag{27.9b}$$

Comparison of Eq. (27.9) with Eq. (27.4) indicates that at this point, the solution has been reduced to an eigenvalue problem.

EXAMPLE 27.4 Eigenvalues and Eigenvectors for a Mass-Spring System

Problem Statement. Evaluate the eigenvalues and the eigenvectors of Eq. (27.9) for the case where $m_1 = m_2 = 40$ kg and $k = 200$ N/m.

Solution. Substituting the parameter values into Eq. (27.9) yields

$$(10 - \omega^2) A_1 - 5A_2 = 0$$
$$-5A_1 + (10 - \omega^2) A_2 = 0$$

The determinant of this system is [recall Eq. (9.3)]

$$(\omega^2)^2 - 20\omega^2 + 75 = 0$$

which can be solved by the quadratic formula for $\omega^2 = 15$ and 5 s^{-2}. Therefore, the frequencies for the vibrations of the masses are $\omega = 3.873$ s^{-1} and 2.236 s^{-1}, respectively. These values can be used to determine the periods for the vibrations with Eq. (27.7). For the first mode, $T_p = 1.62$ s, and for the second, $T_p = 2.81$ s.

As stated in Sec. 27.2.1, a unique set of values cannot be obtained for the unknowns. However, their ratios can be specified by substituting the eigenvalues back into the equations. For example, for the first mode ($\omega^2 = 15$ s^{-2}), $A_1 = -A_2$. For the second mode ($\omega^2 = 5$ s^{-2}), $A_1 = A_2$.

This example provides valuable information regarding the behavior of the system in Fig. 27.5. Aside from its period, we know that if the system is vibrating in the first mode, the amplitude of the second mass will be equal but of opposite sign to the amplitude of the first. As in Fig. 27.6*a*, the masses vibrate apart and then together indefinitely.

In the second mode, the two masses have equal amplitudes at all times. Thus, as in Fig. 27.6*b*, they vibrate back and forth in unison. It should be noted that the configuration of the amplitudes provides guidance on how to set their initial values to attain pure motion

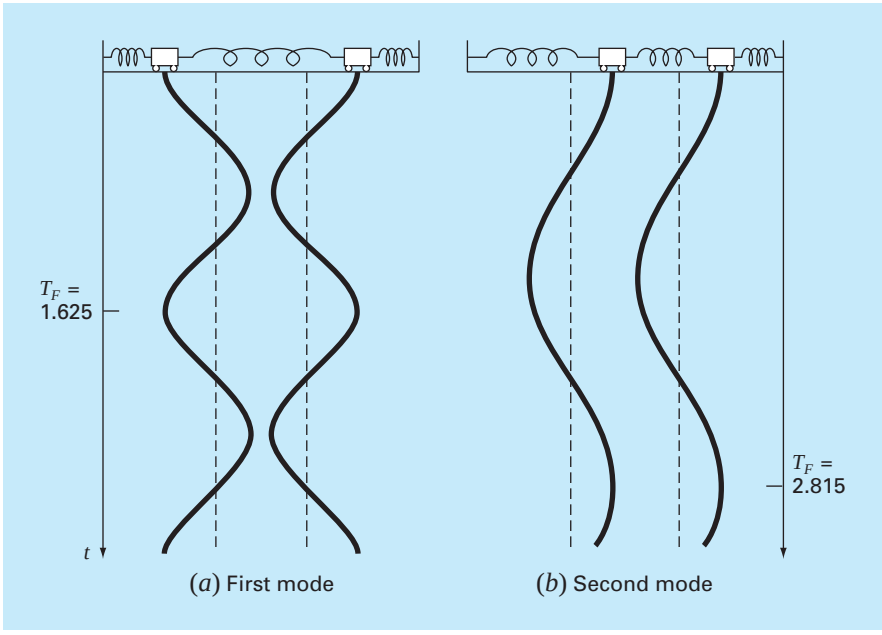

FIGURE 27.6
The principal modes of vibration of two equal masses connected by three identical springs between fixed walls.

in either of the two modes. Any other configuration will lead to superposition of the modes (recall Chap. 19).

27.2.3 A Boundary-Value Problem

Now that you have been introduced to eigenvalues, we turn to the type of problem that is the subject of the present chapter: boundary-value problems for ordinary differential equations. Figure 27.7 shows a physical system that can serve as a context for examining this type of problem.

The curvature of a slender column subject to an axial load P can be modeled by

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{27.10}$$

where d^2y/dx^2 specifies the curvature, $M =$ the bending moment, $E =$ the modulus of elasticity, and $I =$ the moment of inertia of the cross section about its neutral axis. Considering the free body in Fig. 27.7*b*, it is clear that the bending moment at x is $M = -Py$. Substituting this value into Eq. (27.10) gives

$$\frac{d^2y}{dx^2} + p^2y = 0 \tag{27.11}$$

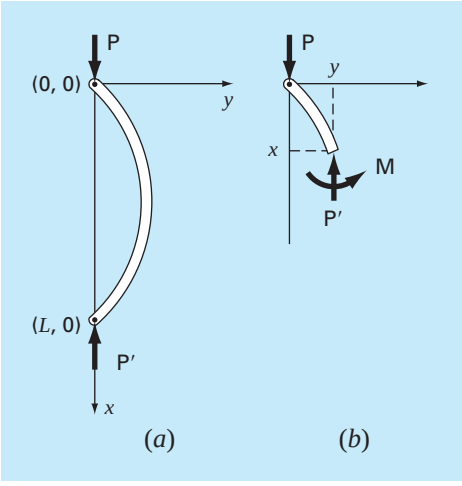

FIGURE 27.7
(*a*) A slender rod. (*b*) A free-body diagram of a rod.

where

$$p^2 = \frac{P}{EI} \tag{27.12}$$

For the system in Fig. 27.7, subject to the boundary conditions

$$y(0) = 0 \tag{27.13a}$$

$$y(L) = 0 \tag{27.13b}$$

the general solution for Eq. (27.11) is

$$y = A\sin(px) + B\cos(px) \tag{27.14}$$

where A and B are arbitrary constants that are to be evaluated via the boundary conditions. According to the first condition [Eq. (27.13*a*)],

$$0 = A\sin(0) + B\cos(0)$$

Therefore, we conclude that $B = 0$.

According to the second condition [Eq. (27.13*b*)],

$$0 = A\sin(pL) + B\cos(pL)$$

But, since $B = 0$, $A\sin(pL) = 0$. Because $A = 0$ represents a trivial solution, we conclude that $\sin(pL) = 0$. For this equality to hold,

$$pL = n\pi \qquad \text{for } n = 1, 2, 3, \ldots \pi \tag{27.15}$$

Thus, there are an infinite number of values that meet the boundary condition. Equation (27.15) can be solved for

$$p = \frac{n\pi}{L} \qquad \text{for } n = 1, 2, 3, \ldots \tag{27.16}$$

which are the eigenvalues for the column.

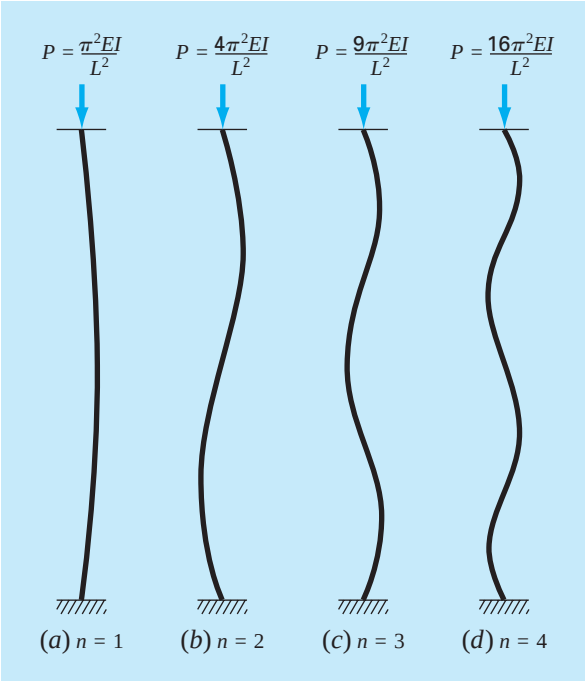

FIGURE 27.8
The first four eigenvalues for the slender rod from Fig. 27.7.

Figure 27.8, which shows the solution for the first four eigenvalues, can provide insight into the physical significance of the results. Each eigenvalue corresponds to a way in which the column buckles. Combining Eqs. (27.12) and (27.16) gives

$$P = \frac{n^2 \pi^2 E I}{L^2} \qquad \text{for } n = 1, 2, 3, \ldots \tag{27.17}$$

These can be thought of as *buckling loads* because they represent the levels at which the column moves into each succeeding buckling configuration. In a practical sense, it is usually the first value that is of interest because failure will usually occur when the column first buckles. Thus, a critical load can be defined as

$$P = \frac{\pi^2 E I}{L^2}$$

which is formally known as *Euler's formula.*

EXAMPLE 27.5 Eigenvalue Analysis of an Axially Loaded Column

Problem Statement. An axially loaded wooden column has the following characteristics: $E = 10 \times 10^9$ Pa, $I = 1.25 \times 10^{-5}$ m^4, and $L = 3$ m. Determine the first eight eigenvalues and the corresponding buckling loads.

Solution. Equations (27.16) and (27.17) can be used to compute

n	p, m^{-2}	P, kN
1	1.0472	137.078
2	2.0944	548.311
3	3.1416	1233.701
4	4.1888	2193.245
5	5.2360	3426.946
6	6.2832	4934.802
7	7.3304	6716.814
8	8.3776	8772.982

The critical buckling load is, therefore, 137.078 kN.

Although analytical solutions of the sort obtained above are useful, they are often difficult or impossible to obtain. This is usually true when dealing with complicated systems or those with heterogeneous properties. In such cases, numerical methods of the sort described next are the only practical alternative.

27.2.4 The Polynomial Method

Equation (27.11) can be solved numerically by substituting a central finite-divided-difference approximation (Fig. 23.3) for the second derivative to give

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p^2 y_i = 0$$

which can be expressed as

$$y_{i-1} - (2 - h^2 p^2) y_i + y_{i+1} = 0 \tag{27.18}$$

Writing this equation for a series of nodes along the axis of the column yields a homogeneous system of equations. For example, if the column is divided into five segments (that is, four interior nodes), the result is

$$\begin{bmatrix} (2-h^2p^2) & -1 & 0 & 0 \\ -1 & (2-h^2p^2) & -1 & 0 \\ 0 & -1 & (2-h^2p^2) & -1 \\ 0 & 0 & -1 & (2-h^2p^2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = 0 \tag{27.19}$$

Expansion of the determinant of the system yields a polynomial, the roots of which are the eigenvalues. This approach, called the *polynomial method*, is performed in the following example.

EXAMPLE 27.6 The Polynomial Method

Problem Statement. Employ the polynomial method to determine the eigenvalues for the axially loaded column from Example 27.5 using (*a*) one, (*b*) two, (*c*) three, and (*d*) four interior nodes.

Solution.

(**a**) Writing Eq. (27.18) for one interior node yields ($h = 3/2$)

$$-(2 - 2.25p^2)y_1 = 0$$

Thus, for this simple case, the eigenvalue is analyzed by setting the determinant equal to zero

$$2 - 2.25p^2 = 0$$

and solving for $p = \pm 0.9428$, which is about 10 percent less than the exact value of 1.0472 obtained in Example 27.4.

(**b**) For two interior nodes ($h = 3/3$), Eq. (27.18) is written as

$$\begin{bmatrix} (2-p^2) & -1 \\ -1 & (2-p^2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = 0$$

Expansion of the determinant gives

$$(2 - p^2)^2 - 1 = 0$$

which can be solved for $p = \pm 1$ and ± 1.73205. Thus, the first eigenvalue is now about 4.5 percent low and a second eigenvalue is obtained that is about 17 percent low.

(**c**) For three interior points ($h = 3/4$), Eq. (27.18) yields

$$\begin{bmatrix} 2-0.5625p^2 & -1 & 0 \\ -1 & 2-0.5625p^2 & -1 \\ 0 & -1 & 2-0.5625p^2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = 0 \qquad \text{(E27.6.1)}$$

The determinant can be set equal to zero and expanded to give

$$(2 - 0.5625p^2)^3 - 2(2 - 0.5625p^2) = 0$$

For this equation to hold, $2 - 0.5625p^2 = 0$ and $2 - 0.5625p^2 = \sqrt{2}$. Therefore, the first three eigenvalues can be determined as

$$p = \pm 1.0205 \qquad |\varepsilon_t| = 2.5\%$$
$$p = \pm 1.8856 \qquad |\varepsilon_t| = 10\%$$
$$p = \pm 2.4637 \qquad |\varepsilon_t| = 22\%$$

(**d**) For four interior points ($h = 3/5$), the result is Eq. (27.19) with $2 - 0.36p^2$ on the diagonal. Setting the determinant equal to zero and expanding it gives

$$(2 - 0.36p^2)^4 - 3(2 - 0.36p^2)^2 + 1 = 0$$

which can be solved for the first four eigenvalues

$$p = \pm 1.0301 \qquad |\varepsilon_t| = 1.6\%$$
$$p = \pm 1.9593 \qquad |\varepsilon_t| = 6.5\%$$
$$p = \pm 2.6967 \qquad |\varepsilon_t| = 14\%$$
$$p = \pm 3.1702 \qquad |\varepsilon_t| = 24\%$$

TABLE 27.2 The results of applying the polynomial method to an axially loaded column. The numbers in parentheses represent the absolute value of the true percent relative error.

		Polynomial Method			
Eigenvalue	**True**	***h* = 3/2**	***h* = 3/3**	***h* = 3/4**	***h* = 3/5**
1	1.0472	0.9428 (10%)	1.0000 (4.5%)	1.0205 (2.5%)	1.0301 (1.6%)
2	2.0944		1.7321 (21%)	1.8856 (10%)	1.9593 (65%)
3	3.1416			2.4637 (22%)	2.6967 (14%)
4	4.1888				3.1702 (24%)

Table 27.2, which summarizes the results of this example, illustrates some fundamental aspects of the polynomial method. As the segmentation is made more refined, additional eigenvalues are determined and the previously determined values become progressively more accurate. Thus, the approach is best suited for cases where the lower eigenvalues are required.

27.2.5 The Power Method

The *power method* is an iterative approach that can be employed to determine the largest eigenvalue. With slight modification, it can also be employed to determine the smallest and the intermediate values. It has the additional benefit that the corresponding eigenvector is obtained as a by-product of the method.

Determination of the Largest Eigenvalue. To implement the power method, the system being analyzed must be expressed in the form

$$[A]\{X\} = \lambda\{X\} \tag{27.20}$$

As illustrated by the following example, Eq. (27.20) forms the basis for an iterative solution technique that eventually yields the highest eigenvalue and its associated eigenvector.

EXAMPLE 27.7 Power Method for Highest Eigenvalue

Problem Statement. Employ the power method to determine the highest eigenvalue for part **(c)** of Example 27.6.

Solution. The system is first written in the form of Eq. (27.20),

$$\begin{aligned} 3.556x_1 - 1.778x_2 &= \lambda x_1 \\ -1.778x_1 + 3.556x_2 - 1.778x_3 &= \lambda x_2 \\ -1.778x_2 + 3.556x_3 &= \lambda x_3 \end{aligned}$$

Then, assuming the x's on the left-hand side of the equation are equal to 1,

$$\begin{aligned} 3.556(1) - 1.778(1) &= 1.778 \\ -1.778(1) + 3.556(1) - 1.778(1) &= 0 \\ -1.778(1) + 3.556(1) &= 1.778 \end{aligned}$$

Next, the right-hand side is normalized by 1.778 to make the largest element equal to

$$\begin{Bmatrix} 1.778 \\ 0 \\ 1.778 \end{Bmatrix} = 1.778 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

Thus, the first estimate of the eigenvalue is 1.778. This iteration can be expressed concisely in matrix form as

$$\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1.778 \\ 0 \\ 1.778 \end{Bmatrix} = 1.778 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

The next iteration consists of multiplying $[A]$ by $\lfloor 1 \quad 0 \quad 1 \rfloor^T$ to give

$$\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 3.556 \\ -3.556 \\ 3.556 \end{Bmatrix} = 3.556 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

Therefore, the eigenvalue estimate for the second iteration is 3.556, which can be employed to determine the error estimate

$$|\varepsilon_a| = \left| \frac{3.556 - 1.778}{3.556} \right| 100\% = 50\%$$

The process can then be repeated.

Third iteration:

$$\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 5.334 \\ -7.112 \\ 5.334 \end{Bmatrix} = -7.112 \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix}$$

where $|\varepsilon_a| = 150\%$ (which is high because of the sign change).

Fourth iteration:

$$\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix} \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix} = \begin{Bmatrix} -4.445 \\ 6.223 \\ -4.445 \end{Bmatrix} = 6.223 \begin{Bmatrix} -0.714 \\ 1 \\ -0.714 \end{Bmatrix}$$

where $|\varepsilon_a| = 214\%$ (again inflated because of sign change).

Fifth iteration:

$$\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix} \begin{Bmatrix} -0.714 \\ 1 \\ -0.714 \end{Bmatrix} = \begin{Bmatrix} -4.317 \\ 6.095 \\ -4.317 \end{Bmatrix} = 6.095 \begin{Bmatrix} -0.708 \\ 1 \\ -0.708 \end{Bmatrix}$$

Thus, the normalizing factor is converging on the value of 6.070 ($= 2.4637^2$) obtained in part **(c)** of Example 27.6.

Note that there are some instances where the power method will converge to the second-largest eigenvalue instead of to the largest. James, Smith, and Wolford (1985) provide an illustration of such a case. Other special cases are discussed in Fadeev and Fadeeva (1963).

Determination of the Smallest Eigenvalue. There are often cases in engineering where we are interested in determining the smallest eigenvalue. Such was the case for the rod in Fig. 27.7, where the smallest eigenvalue could be used to identify a critical buckling load. This can be done by applying the power method to the matrix inverse of $[A]$. For this case, the power method will converge on the largest value of $1/\lambda$—in other words, the smallest value of λ.

EXAMPLE 27.8 Power Method for Lowest Eigenvalue

Problem Statement. Employ the power method to determine the lowest eigenvalue for part **(c)** of Example 27.6.

Solution. After dividing Eq. E27.6.1 by h^2 ($= 0.5625$), its matrix inverse can be evaluated as

$$[A]^{-1} = \begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix}$$

Using the same format as in Example 27.9, the power method can be applied to this matrix.

First iteration:

$$\begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.884 \\ 1.124 \\ 0.884 \end{Bmatrix} = 1.124 \begin{Bmatrix} 0.751 \\ 1 \\ 0.751 \end{Bmatrix}$$

Second iteration:

$$\begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix} \begin{Bmatrix} 0.751 \\ 1 \\ 0.751 \end{Bmatrix} = \begin{Bmatrix} 0.704 \\ 0.984 \\ 0.704 \end{Bmatrix} = 0.984 \begin{Bmatrix} 0.715 \\ 1 \\ 0.715 \end{Bmatrix}$$

where $|\varepsilon_a| = 14.6\%$.

Third iteration:

$$\begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix} \begin{Bmatrix} 0.715 \\ 1 \\ 0.715 \end{Bmatrix} = \begin{Bmatrix} 0.684 \\ 0.964 \\ 0.684 \end{Bmatrix} = 0.964 \begin{Bmatrix} 0.709 \\ 1 \\ 0.709 \end{Bmatrix}$$

where $|\varepsilon_a| = 4\%$.

Thus, after only three iterations, the result is converging on the value of 0.9602, which is the reciprocal of the smallest eigenvalue, 1.0205($= \sqrt{1/0.9602}$), obtained in Example 27.6c.

Determination of Intermediate Eigenvalues. After finding the largest eigenvalue, it is possible to determine the next highest by replacing the original matrix by one that includes only the remaining eigenvalues. The process of removing the largest known eigenvalue is called *deflation.* The technique outlined here, *Hotelling's method,* is designed for symmetric matrices. This is because it exploits the orthogonality of the eigenvectors of such matrices, which can be expressed as

$$\{X\}_i^T\{X\}_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \tag{27.21}$$

where the components of the eigenvector $\{X\}$ have been normalized so that $\{X\}^T\{X\} = 1$, that is, so that the sum of the squares of the components equals 1. This can be accomplished by dividing each of the elements by the normalizing factor

$$\sqrt{\sum_{k=1}^{n} x_k^2}$$

Now, a new matrix $[A]_2$ can be computed as

$$[A]_2 = [A]_1 - \lambda_1\{X\}_1\{X\}_1^T \tag{27.22}$$

where $[A]_1$ = the original matrix and λ_1 = the largest eigenvalue. If the power method is applied to this matrix, the iteration process will converge to the second largest eigenvalue, λ_2. To show this, first postmultiply Eq. (27.22) by $\{X\}_1$,

$$[A]_2\{X\}_1 = [A]_1\{X\}_1 - \lambda_1\{X\}_1\{X\}_1^T\{X\}_1$$

Invoking the orthogonality principle converts this equation to

$$[A]_2\{X\}_1 = [A]_1\{X\}_1 - \lambda_1\{X\}_1$$

where the right-hand side is equal to zero according to Eq. (27.20). Thus, $[A]_2\{X\}_1 = 0$. Consequently, $\lambda = 0$ and $\{X\} = \{X\}_1$ is a solution to $[A]_2\{X\} = \lambda\{X\}$. In other words, the $[A]_2$ has eigenvalues of $0, \lambda_2, \lambda_3, \ldots, \lambda_n$. The largest eigenvalue, λ_1, has been replaced by a 0 and, therefore, the power method will converge on the next biggest λ_2.

The above process can be repeated by generating a new matrix $[A]_3$, etc. Although in theory this process could be continued to determine the remaining eigenvalues, it is limited by the fact that errors in the eigenvectors are passed along at each step. Thus, it is only of value in determining several of the highest eigenvalues. Although this is somewhat of a shortcoming, such information is precisely what is required in many engineering problems.

27.2.6 Other Methods

A wide variety of additional methods are available for solving eigenvalue problems. Most are based on a two-step process. The first step involves transforming the original matrix to a simpler form (for example, tridiagonal) that retains all the original eigenvalues. Then, iterative methods are used to determine these eigenvalues.

Many of these approaches are designed for special types of matrices. In particular, a variety of techniques are devoted to symmetric systems. For example, *Jacobi's method*

transforms a symmetric matrix to a diagonal matrix by eliminating off-diagonal terms in a systematic fashion. Unfortunately, the method requires an infinite number of operations because the removal of each nonzero element often creates a new nonzero value at a previous zero element. Although an infinite time is required to create all nonzero off-diagonal elements, the matrix will eventually tend toward a diagonal form. Thus, the approach is iterative in that it is repeated until the off-diagonal terms are "sufficiently" small.

Given's method also involves transforming a symmetric matrix into a simpler form. However, in contrast to the Jacobi method, the simpler form is tridiagonal. In addition, it differs in that the zeros that are created in off-diagonal positions are retained. Consequently, it is finite and, thus, more efficient than Jacobi's method.

Householder's method also transforms a symmetric matrix into a tridiagonal form. It is a finite method and is more efficient than Given's approach in that it reduces whole rows and columns of off-diagonal elements to zero.

Once a tridiagonal system is obtained from Given's or Householder's method, the remaining step involves finding the eigenvalues. A direct way to do this is to expand the determinant. The result is a sequence of polynomials that can be evaluated iteratively for the eigenvalues.

Aside from symmetric matrices, there are also techniques that are available when all eigenvalues of a general matrix are required. These include the *LR method* of Rutishauser and the *QR method* of Francis. Although the QR method is less efficient, it is usually the preferred approach because it is more stable. As such, it is considered to be the best general-purpose solution method.

Finally, it should be mentioned that the aforementioned techniques are often used in tandem to capitalize on their respective strengths. For example, Given's and Householder's methods can also be applied to nonsymmetric systems. The result will not be tridiagonal but rather a special type called the *Hessenberg form.* One approach is to exploit the speed of Householder's approach by employing it to transform the matrix to this form and then use the stable QR algorithm to find the eigenvalues. Additional information on these and other issues related to eigenvalues can be found in Ralston and Rabinowitz (1978), Wilkinson (1965), Fadeev and Fadeeva (1963), and Householder (1953, 1964). Computer codes can be found in a number of sources including Press et al. (1992). Rice (1983) discusses available software packages.

SOFTWARE

27.3 ODES AND EIGENVALUES WITH SOFTWARE PACKAGES

Software packages have great capabilities for solving ODEs and determining eigenvalues. This section outlines some of the ways in which they can be applied for this purpose.

27.3.1 Excel

Excel's direct capabilities for solving eigenvalue problems and ODEs are limited. However, if some programming is done (for example, macros), they can be combined with Excel's visualization and optimization tools to implement some interesting applications. Section 28.1 provides an example of how the Excel Solver can be used for parameter estimation of an ODE.