CHAPTER 27

Boundary-Value and Eigenvalue Problems

Recall from our discussion at the beginning of Part Seven that an ordinary differential equation is accompanied by auxiliary conditions. These conditions are used to evaluate the constants of integration that result during the solution of the equation. For an nth-order equation, n conditions are required. If all the conditions are specified at the same value of the independent variable, then we are dealing with an *initial-value problem* (Fig. 27.1*a*). To this point, the material in Part Seven has been devoted to this type of problem.

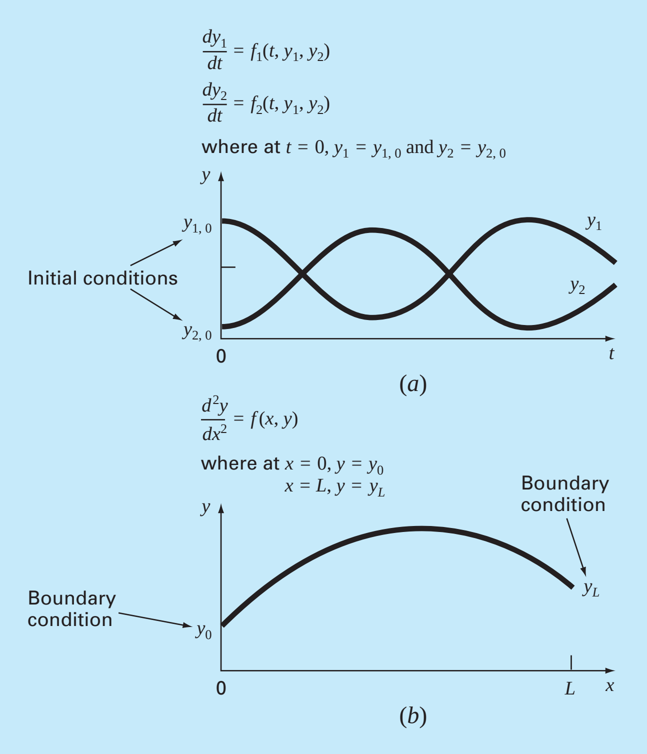


FIGURE 27.1
Initial-value versus boundary-value problems. (*a*) An initial-value problem where all the conditions are specified at the same value of the independent variable. (*b*) A boundary-value problem where the conditions are specified at different values of the independent variable.

In contrast, there is another application for which the conditions are not known at a single point, but rather, are known at different values of the independent variable. Because these values are often specified at the extreme points or boundaries of a system, they are customarily referred to as *boundary-value problems* (Fig. 27.1*b*). A variety of significant engineering applications fall within this class. In this chapter, we discuss two general approaches for obtaining their solution: the shooting method and the finite-difference approach. Additionally, we present techniques to approach a special type of boundary-value problem: the determination of eigenvalues. Of course, eigenvalues also have many applications beyond those involving boundary-value problems.

27.1 GENERAL METHODS FOR BOUNDARY-VALUE PROBLEMS

The conservation of heat can be used to develop a heat balance for a long, thin rod (Fig. 27.2). If the rod is not insulated along its length and the system is at a steady state, the equation that results is

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0 \tag{27.1}$$

where h' is a heat transfer coefficient (m^{-2}) that parameterizes the rate of heat dissipation to the surrounding air and T_a is the temperature of the surrounding air (°C).

To obtain a solution for Eq. (27.1), there must be appropriate boundary conditions. A simple case is where the temperatures at the ends of the bar are held at fixed values. These can be expressed mathematically as

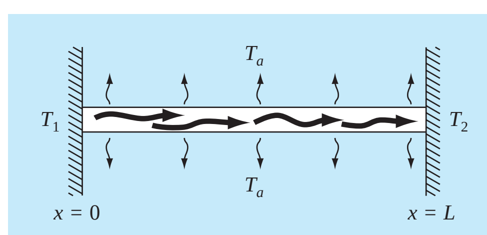
$$T(0) = T_1$$
$$T(L) = T_2$$

With these conditions. Eq. (27.1) can be solved analytically using calculus. For a 10-m rod with $T_a = 20$, $T_1 = 40$, $T_2 = 200$, and $h' = 0.01$, the solution is

$$T = 73.4523e^{0.1x} - 53.4523e^{-0.1x} + 20 \tag{27.2}$$

In the following sections, the same problem will be solved using numerical approaches.

FIGURE 27.2
A noninsulated uniform rod positioned between two bodies of constant but different temperature. For this case $T_1 > T_2$ and $T_2 > T_a$.



27.1.1 The Shooting Method

The *shooting method* is based on converting the boundary-value problem into an equivalent initial-value problem. A trial-and-error approach is then implemented to solve the initial-value version. The approach can be illustrated by an example.

EXAMPLE 27.1

The Shooting Method

Problem Statement. Use the shooting method to solve Eq. (27.1) for a 10-m rod with $h' = 0.01\ \text{m}^{-2}$, $T_a = 20$, and the boundary conditions

$$T(0) = 40 \qquad T(10) = 200$$

Solution. Using the same approach as was employed to transform Eq. (PT7.2) into Eqs. (PT7.3) through (PT7.6), the second-order equation can be expressed as two first-order ODEs:

$$\frac{dT}{dx} = z \tag{E27.1.1}$$

$$\frac{dz}{dx} = h'(T - T_a) \tag{E27.1.2}$$

To solve these equations, we require an initial value for z. For the shooting method, we guess a value—say, $z(0) = 10$. The solution is then obtained by integrating Eq. (E27.1.1) and (E27.1.2) simultaneously. For example, using a fourth-order RK method with a step size of 2, we obtain a value at the end of the interval of $T(10) = 168.3797$ (Fig. 27.3*a*), which differs from the boundary condition of $T(10) = 200$. Therefore, we make another guess, $z(0) = 20$, and perform the computation again. This time, the result of $T(10) = 285.8980$ is obtained (Fig. 27.3*b*).

Now, because the original ODE is linear, the values

$$z(0) = 10 \qquad T(10) = 168.3797$$

and

$$z(0) = 20 \qquad T(10) = 285.8980$$

are linearly related. As such, they can be used to compute the value of $z(0)$ that yields $T(10) = 200$. A linear interpolation formula [recall Eq. (18.2)] can be employed for this purpose:

$$z(0) = 10 + \frac{20 - 10}{285.8980 - 168.3797}(200 - 168.3797) = 12.6907$$

This value can then be used to determine the correct solution, as depicted in Fig. 27.3*c*.

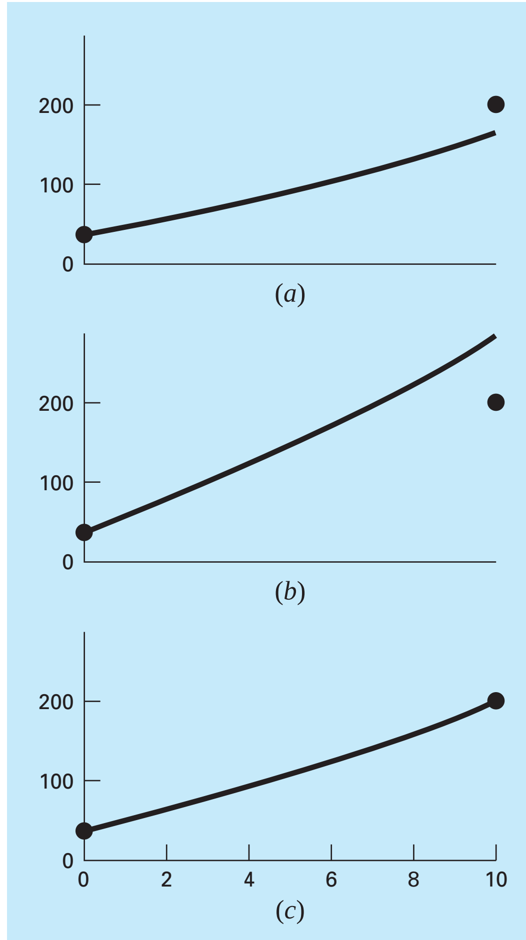


FIGURE 27.3
The shooting method: (*a*) the first "shot," (*b*) the second "shot," and (*c*) the final exact "hit."

Nonlinear Two-Point Problems. For nonlinear boundary-value problems, linear interpolation or extrapolation through two solution points will not necessarily result in an accurate estimate of the required boundary condition to attain an exact solution. An alternative is to perform three applications of the shooting method and use a quadratic interpolating polynomial to estimate the proper boundary condition. However, it is unlikely that such an approach would yield the exact answer, and additional iterations would be necessary to obtain the solution.

Another approach for a nonlinear problem involves recasting it as a roots problem. Recall that the general form of a root problem is to find the value of x that makes the function $f(x) = 0$. Now, let us use Example 27.1 to understand how the shooting method can be recast in this form.

First, recognize that the solution of the pair of differential equations is also a "function" in the sense that we guess a condition at the left-hand end of the bar, z_0, and the integration yields a prediction of the temperature at the right-hand end, T_{10}. Thus, we can think of the integration as

$$T_{10} = f(z_0)$$

That is, it represents a process whereby a guess of z_0 yields a prediction of T_{10}. Viewed in this way, we can see that what we desire is the value of z_0 that yields a specific value of T_{10}. If, as in the example, we desire $T_{10} = 200$, the problem can be posed as

$$200 = f(z_0)$$

By bringing the goal of 200 over to the right-hand side of the equation, we generate a new function, $g(z_0)$, that represents the difference between what we have, $f(z_0)$, and what we want, 200.

$$g(z_0) = f(z_0) - 200$$

If we drive this new function to zero, we will obtain the solution. The next example illustrates the approach.

EXAMPLE 27.2 The Shooting Method for Nonlinear Problems

Problem Statement. Although it served our purposes for proving a simple boundary-value problem, our model for the bar in Eq. (27.1) was not very realistic. For one thing, such a bar would lose heat by mechanisms such as radiation that are nonlinear.

Suppose that the following nonlinear ODE is used to simulate the temperature of the heated bar:

$$\frac{d^2T}{dx^2} + h''(T_a - T)^4 = 0$$

where $h'' = 5 \times 10^{-8}$. Now, although it is still not a very good representation of heat transfer, this equation is straightforward enough to allow us to illustrate how the shooting method can be used to solve a two-point nonlinear boundary-value problem. The remaining problem conditions are as specified in Example 27.1.

Solution. The second-order equation can be expressed as two first-order ODEs:

$$\frac{dT}{dx} = z$$

$$\frac{dz}{dx} = h''(T - T_a)^4$$

Now, these equations can be integrated using any of the methods described in Chaps. 25 and 26. We used the constant step-size version of the fourth-order RK approach described in Chap. 25. We implemented this approach as an Excel macro function written in Visual

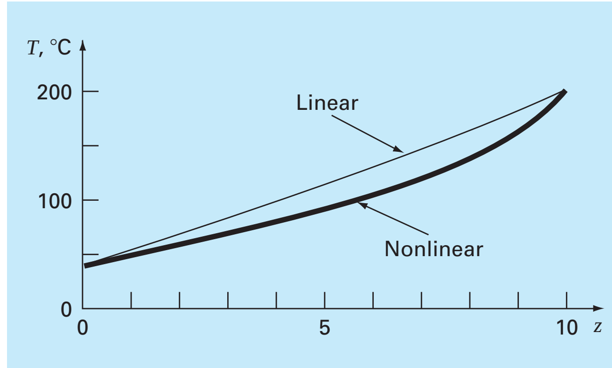


FIGURE 27.4
The result of using the shooting method to solve a nonlinear problem.

BASIC. The function integrated the equations based on an initial guess for $z(0)$ and returned the temperature at $x = 10$. The difference between this value and the goal of 200 was then placed in a spreadsheet cell. The Excel Solver was then invoked to adjust the value of $z(0)$ until the difference was driven to zero.

The result is shown in Fig. 27.4 along with the original linear case. As might be expected, the nonlinear case is curved more than the linear model. This is due to the power of four term in the heat transfer relationship.

The shooting method can become arduous for higher-order equations where the necessity to assume two or more conditions makes the approach somewhat more difficult. For these reasons, alternative methods are available, as described next.

27.1.2 Finite-Difference Methods

The most common alternatives to the shooting method are *finite-difference approaches.* In these techniques, finite divided differences are substituted for the derivatives in the original equation. Thus, a linear differential equation is transformed into a set of simultaneous algebraic equations that can be solved using the methods from Part Three.

For the case of Fig. 27.2, the finite-divided-difference approximation for the second derivative is (recall Fig. 23.3)

$$\frac{d^2T}{dx^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$

This approximation can be substituted into Eq. (27.1) to give

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - h'(T_i - T_a) = 0$$

Collecting terms gives

$$-T_{i-1} + (2 + h'\Delta x^2)T_i - T_{i+1} = h'\Delta x^2 T_a \tag{27.3}$$

This equation applies for each of the interior nodes of the rod. The first and last interior nodes, T_{i-1} and T_{i+1}, respectively, are specified by the boundary conditions. Therefore, the resulting set of linear algebraic equations will be tridiagonal. As such, it can be solved with the efficient algorithms that are available for such systems (Sec. 11.1).

EXAMPLE 27.3 Finite-Difference Approximation of Boundary-Value Problems

Problem Statement. Use the finite-difference approach to solve the same problem as in Example 27.1.

Solution. Employing the parameters in Example 27.1, we can write Eq. (27.3) for the rod from Fig. 27.2. Using four interior nodes with a segment length of $\Delta x = 2$ m results in the following equations:

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix}$$

which can be solved for

$$\{T\}^T = \lfloor 65.9698 \quad 93.7785 \quad 124.5382 \quad 159.4795 \rfloor$$

Table 27.1 provides a comparison between the analytical solution [Eq. (27.2)] and the numerical solutions obtained in Examples 27.1 and 27.3. Note that there are some discrepancies among the approaches. For both numerical methods, these errors can be mitigated by decreasing their respective step sizes. Although both techniques perform well for the present case, the finite-difference approach is preferred because of the ease with which it can be extended to more complex cases.

The fixed (or *Dirichlet*) boundary condition used in the previous example is but one of several types that are commonly employed in engineering and science. A common alternative, called the *Neumann boundary condition*, is the case where the derivative is given.

TABLE 27.1 Comparison of the exact analytical solution with the shooting and finite-difference methods.

x	True	Shooting Method	Finite Difference
0	40	40	40
2	65.9518	65.9520	65.9698
4	93.7478	93.7481	93.7785
6	124.5036	124.5039	124.5382
8	159.4534	159.4538	159.4795
10	200	200	200

We can use the heated rod model to demonstrate how derivative boundary condition can be incorporated into the finite-difference approach,

$$0 = \frac{d^2T}{dx^2} + h'(T_\infty - T)$$

However, in contrast to our previous discussions, we will prescribe a derivative boundary condition at one end of the rod,

$$\frac{dT}{dx}(0) = T_a'$$

$$T(L) = T_b$$

Thus, we have a derivative boundary condition at one end of the solution domain and a fixed boundary condition at the other.

As was done in Example 27.3, the rod is divided into a series of nodes and a finite-difference version of the differential equation (Eq. 27.3) is applied to each interior node. However, because its temperature is not specified, the node at the left end must also be included. Writing Eq. (27.3) for this node gives

$$-T_{-1} + (2 + h'\Delta x^2)T_0 - T_1 = h'\Delta x^2 T_\infty \qquad (27.3a)$$

Notice that an imaginary node (-1) lying to the left of the rod's end is required for this equation. Although this exterior point might seem to represent a difficulty, it actually serves as the vehicle for incorporating the derivative boundary condition into the problem. This is done by representing the first derivative in the x dimension at (0) by the centered difference

$$\frac{dT}{dx} = \frac{T_1 - T_{-1}}{2\Delta x}$$

which can be solved for

$$T_{-1} = T_1 - 2\Delta x \frac{dT}{dx}$$

Now we have a formula for T_{-1} that actually reflects the impact of the derivative. It can be substituted into Eq. (27.3*a*) to give

$$(2 + h'\Delta x^2)T_0 - 2T_1 = h'\Delta x^2 T_\infty - 2\Delta x \frac{dT}{dx} \qquad (27.3b)$$

Consequently, we have incorporated the derivative into the balance.

A common example of a derivative boundary condition is the situation where the end of the rod is insulated. In this case, the derivative is set to zero. This conclusion follows directly from Fourier's law, which states that the heat flux is directly proportional to the temperature gradient. Thus, insulating a boundary means that the heat flux (and consequently the gradient) must be zero.

Aside from the shooting and finite-difference methods, there are other techniques available for solving boundary-value problems. Some of these will be described in Part Eight. These include steady-state (Chap. 29) and transient (Chap. 30) solution of two-dimensional