A Summary of Solutions to the Convective Diffusion Equation

General Governing Equation for Homogeneous, Anisotropic Turbulence, Steady Velocity U:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = E_x\frac{\partial^2 c}{\partial x^2} + E_y\frac{\partial^2 c}{\partial y^2} + E_z\frac{\partial^2 c}{\partial z^2} - Kc \tag{0-1}$$

1 Instantaneous Point Sources (IPtS)

1.1 IPtS at (x_1, y_1, z_1), steady unidirectional velocity field $\vec{q}$ = (U, 0, 0)

$$c = \frac{M}{(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}} \quad \exp -\left\{\frac{[(x-x_1)-Ut]^2}{4E_x t} + \frac{(y-y_1)^2}{4E_y t} + \frac{(z-z_1)^2}{4E_z t} + Kt\right\} \tag{1-1}$$

where M = mass imput.

1.2 Fluid at rest, isotropic, no decay, molecular diffusion

$$c = \frac{M}{(4\pi Dt)^{3/2}} \quad \exp -\left[\frac{r^2}{4Dt}\right] \tag{1-2}$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (x_1 = y_1 = z_1 = 0).$$

1.3 IPtS at (x_1, y_1, z_1) with image at (x_1, y_1, -z_1), Fluid at rest

$$\begin{aligned} c = {} & \frac{M}{(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}} \quad \exp -\left\{\frac{(x-x_1)^2}{4E_x t} + \frac{(y-y_1)^2}{4E_y t} + \frac{(z-z_1)^2}{4E_z t}\right\} \\ & + \frac{M}{(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}} \quad \exp -\left\{\frac{(x-x_1)^2}{4E_x t} + \frac{(y-y_1)^2}{4E_y t} + \frac{(z+z_1)^2}{4E_z t}\right\} \end{aligned} \tag{1-3}$$

1.4 IPtS in Shear Flow with Velocity Field $\vec{q}$ = ([$u_0(t) + \lambda_y y + \lambda_z z$], 0, 0)

where $\lambda_y = \frac{\partial u}{\partial y}$, $\lambda_z = \frac{\partial u}{\partial z}$ = velocity gradients (Okubo and Karweit, 1969).

$$c(t,x,y,z) = \frac{M}{(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}(1+\Phi^2 t^2)^{1/2}} \cdot \exp - \left[\frac{\left\{ x - \int_0^t u_0(t')dt' - \frac{1}{2}\left(\lambda_y y + \lambda_z z\right)t \right\}^2}{4E_x t\left(1+\Phi^2 t^2\right)} + \frac{y^2}{4E_y t} + \frac{z^2}{4E_z t} + Kt \right] \qquad (1\text{-}4)$$

where $\Phi^2 = \frac{1}{12}\left(\lambda_y{}^2 \frac{E_y}{E_x} + \lambda_z{}^2 \frac{E_z}{E_x}\right)$.

2 <u>Instantaneous Line Sources (ILS)</u>

2.1 Steady velocity field $\vec{q}$ = (U, 0, 0) Line source along entire line: x_1, y_1, $-\infty < z < +\infty$

$$c = \frac{m'}{4\pi\left(E_x E_y\right)^{1/2} t} \exp - \left[\frac{\left[\left(x - x_1\right) - Ut\right]^2}{4E_x t} + \frac{\left(y - y_1\right)^2}{4E_y t} + Kt \right] \qquad (2\text{-}1)$$

where m' = mass input per unit length.

2.2 Line source on z - axis between $-Z_2 < z < +Z_2$, $x_1 = y_1 = 0$

$$c = \frac{m'}{8\pi\left(E_x E_y\right)^{1/2} t} \left[\operatorname{erf} \frac{z + Z_2}{\left(4E_z t\right)^{1/2}} - \operatorname{erf} \frac{z - Z_2}{\left(4E_z t\right)^{1/2}} \right] \exp - \left[\frac{\left[x - Ut\right]^2}{4E_x t} + \frac{y^2}{4E_x t} + Kt \right] \qquad (2\text{-}2)$$

3 Instantaneous Plane Source (IPlS)

Injection plane at x_1

$$c = \frac{m''}{\sqrt{4\pi E_x t}} \quad \exp - \left[\frac{\left[(x - x_1) - Ut\right]^2}{4E_x t} + Kt \right] \tag{3-1}$$

where m'' = mass input per unit area

4 Continous Point Source (CPtS)

Time integration of instantaneous sources

$$c = d \int_0^{t_1} \frac{1}{\left(t - \tau\right)^{3/2}} \quad \exp - \left[\frac{a}{(t - \tau)} + b(t - \tau) \right] d\tau \tag{4-1}$$

where

$$a = \frac{\left(x - x_1\right)^2}{4E_x} + \frac{\left(y - y_1\right)^2}{4E_y} + \frac{\left(z - z_1\right)^2}{4E_z}$$

$$b = \frac{U^2}{4E_x} + K$$

$$d = \frac{q \exp\left[\frac{\left(x - x_1\right)U}{2E_x} \right]}{\left(4\pi\right)^{3/2} \left(E_x E_y E_z\right)^{1/2}}$$

where q = time rate of mass injection = $\frac{dM}{d\tau}$

4.1 Solution $t > t_1$ where t_1 = duration of injection

$$c = \frac{d\sqrt{\pi}}{2\sqrt{a}} \left\{ e^{2\sqrt{ab}} \left[\operatorname{erf}\left(\sqrt{\frac{a}{(t - t_1)}} + \sqrt{b(t - t_1)} \right) - \operatorname{erf}\left(\sqrt{\frac{a}{t}} + \sqrt{bt} \right) \right] \quad + \right.$$
$$\left. e^{-2\sqrt{ab}} \left[\operatorname{erf}\left(\sqrt{\frac{a}{(t - t_1)}} - \sqrt{b(t - t_1)} \right) - \operatorname{erf}\left(\sqrt{\frac{a}{t}} - \sqrt{bt} \right) \right] \right\} \tag{4-2}$$

4.2 Continuous injection $t_1 = t$

$$c = \frac{d\sqrt{\pi}}{2\sqrt{a}}\left\{e^{2\sqrt{ab}}\operatorname{erfc}\left[\left(\sqrt{\frac{a}{t}}+\sqrt{bt}\right)\right]+e^{-2\sqrt{ab}}\operatorname{erfc}\left[\left(\sqrt{\frac{a}{t}}-\sqrt{bt}\right)\right]\right\} \tag{4-3}$$

Steady-state solution $t \to \infty$

$$\bar{c} = \frac{d\sqrt{\pi}}{\sqrt{a}}\exp-\left[2\sqrt{ab}\right] \tag{4-4}$$

Special case: Steady, homogenous, isotropic

$$\bar{c} = \frac{q}{4\pi Er}\exp-\left[\frac{r\sqrt{U^2+4EK}-xU}{2E}\right] \tag{4-5}$$

where, $r = \sqrt{x^2+y^2+z^2}$

4.3 CPtS (continuous point source) neglecting longitudinal diffusion

Governing equation, homogenous turbulence

$$U\frac{\partial\bar{c}}{\partial x} = E_y\frac{\partial^2\bar{c}}{\partial y^2}+E_z\frac{\partial^2\bar{c}}{\partial z^2}-K\bar{c} \tag{4-6}$$

Solution

$$\bar{c} \cong \frac{q}{4\pi\left(E_yE_z\right)^{1/2}x}\exp-\left[\frac{y^2U}{4xE_y}+\frac{z^2U}{4xE_z}+\frac{Kx}{U}\right] \tag{4-7}$$

4.4 CPtS in uniform flow with anisotropic, nonhomogenous turbulence

Governing equation

$$U\frac{\partial\bar{c}}{\partial x} = E_y(x)\frac{\partial^2\bar{c}}{\partial y^2}+E_z(x)\frac{\partial^2\bar{c}}{\partial z^2} \tag{4-8}$$

Diffusivities: (Walters, 1962)

$$E_y = a_y x^{\alpha} \tag{4-9}$$

$$E_z = a_z x^{\beta}$$

Solution

$$\bar{c} = \frac{q}{2\pi}\sqrt{\frac{(1+\alpha)(1+\beta)}{a_y a_z}}\; x^{-\left(1+\frac{\alpha+\beta}{2}\right)} \left\{ \exp -\left[\frac{(1+\alpha)U}{4x^{(1+\alpha)}}\frac{y^2}{a_y} + \frac{(1+\beta)U}{4x^{(1+\beta)}}\frac{z^2}{a_z} \right] \right\} \qquad (4\text{-}10)$$

4.5 CPtS in Shear Flow $u(z) = a_0 z^{\mu}$ with Non-homogenous, isotropic turbulence

Governing equation (Smith, 1957)

$$u(z)\frac{\partial \bar{c}}{\partial x} = E(z)\frac{\partial^2 \bar{c}}{dy^2} + \frac{\partial}{\partial z}\left(E(z)\frac{\partial \bar{c}}{\partial z} \right) \qquad (4\text{-}11)$$

Solution for $\mu = ½$, $E_z(z) = b_0 z^{1-\mu}$

$$\bar{c} = \frac{q a_0^{\;1/4}}{2\sqrt{3\pi}\,(b_0 x)^{5/4}} \quad \exp - \left[\frac{a_0\left(y^2 + z^2\right)}{4 b_0 x} \right] \qquad (4\text{-}12)$$

5 <u>Continous Line Source (CLS)</u>

$$c = \int_0^t \frac{q'}{4\pi\left(E_x E_y\right)^{1/2}(t-\tau)} \; \exp - \left\{ \frac{\left[x - U(t-\tau)\right]^2}{4E_x(t-\tau)} + \frac{y^2}{4E_y(t-\tau)} + K(t-\tau) \right\} d\tau \qquad (5\text{-}1)$$

where q’ = time rate of mass injection per unit length.

5.1 Steady state solution $\quad t \rightarrow \infty$

$$\bar{c} = \frac{q' e^{\frac{xU}{2E_x}}}{2\pi\left(E_x E_y\right)^{1/2}} \quad K_0(2\beta_2) \qquad (5\text{-}2)$$

where K_0 = modified Bessel funktion of second kind of order zero and

$$\beta_2 = \frac{\sqrt{\left(E_y x^2 + E_x y^2\right)\left(U^2 E_y + 4E_x E_y K\right)}}{4E_x E_y}$$

5.2 CLS, neglecting longitudinal diffusion

$$c \cong \frac{q'}{\sqrt{4\pi x U E_y}} \quad \exp - \left[\frac{y^2 U}{4E_y x} + \frac{xK}{U} \right] \qquad (5\text{-}3)$$

6 Continous Plane Source (CPlS)

Time integration of IPlS

$$c = \int_0^{t_1} \frac{q''}{\sqrt{4\pi E_x (t-\tau)}} \quad \exp - \left\{ \frac{\left[x - U(t-\tau)\right]^2}{4E_x (t-\tau)} + K(t-\tau) \right\} d\tau \tag{6-1}$$

where q'' = time rate of mass injection per unit area.

6.1 Solution for $t \geq t_1$

$$\begin{aligned} c = \frac{q'' e^{\frac{xU}{2E_x}}}{2\Omega} & \left[\left\{ \mathrm{erf}\left(\frac{x + \Omega t}{\sqrt{4E_x t}} \right) - \mathrm{erf}\left[\frac{x + \Omega(t - t_1)}{\sqrt{4E_x (t - t_1)}} \right] \right\} \exp\left(\frac{x\Omega}{2E_x} \right) \right. \\ & \left. - \left\{ \mathrm{erf}\left(\frac{x - \Omega t}{\sqrt{4E_x t}} \right) - \mathrm{erf}\left[\frac{x - \Omega(t - t_1)}{\sqrt{4E_x (t - t_1)}} \right] \right\} \exp\left(\frac{-x\Omega}{2E_x} \right) \right] \end{aligned} \tag{6-2}$$

where $\Omega = \sqrt{U^2 + 4KE_x}$

6.2 Continuous injection $t_1 = t$

$$\begin{aligned} c = \frac{q'' e^{\frac{xU}{2E_x}}}{2\Omega} & \left[\left\{ \mathrm{erf}\left(\frac{x + \Omega t}{\sqrt{4E_x t}} \right) \mp 1 \right\} \exp\left(\frac{x\Omega}{2E_x} \right) \right. \\ & \left. - \left\{ \mathrm{erf}\left(\frac{x - \Omega t}{\sqrt{4E_x t}} \right) \mp 1 \right\} \exp\left(-\frac{x\Omega}{2E_x} \right) \right] \end{aligned} \tag{6-3}$$

Concentration at $x = 0$

$$c_{(x=0)} = \frac{q''}{\Omega} \quad \mathrm{erf}\left(\frac{\Omega t}{\sqrt{4E_x t}} \right) \tag{6-4}$$

Steady-state $t \to \infty$

$$\bar{c} = \frac{q''}{\Omega} \quad \exp\left[\frac{x}{2E_x} \ (U \mp \Omega) \right] \tag{6-5}$$

6.3 CPlS, neglecting longitudinal diffusion in downstream section

$$\bar{c} \cong \frac{q''}{U} \quad \exp -\left[\frac{xK}{U}\right] \quad \text{for } x > 0 \tag{6-6}$$

Neglecting decay in upstream section

$$\bar{c} \cong \frac{q''}{U} \quad \exp\left[\frac{xU}{E_x}\right] \quad \text{for } x < 0 \tag{6-7}$$

7 <u>Continous Plane Source of Limited Extend</u>

7.1 Semi-infinite CPlS

Source over $-\infty < y < 0$, $-\infty < z < +\infty$

Governing equation:

$$U\frac{\partial \bar{c}}{\partial x} = E_y \frac{\partial^2 \bar{c}}{\partial y^2} - Kc \tag{7-1}$$

Solution:

$$\bar{c} = \frac{q''}{2U} e^{-\frac{Kx}{U}} \operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{U}{E_y x}}\right) \tag{7-2}$$

7.2 Rectangular CPlS Brooks (1960)

$$\text{Source over} -\frac{b}{2} < y < +\frac{b}{2} \;, \quad -\infty < z < +\infty$$

7.2.1 Homogenous turbulence E_y = const.

$$\frac{\bar{c}}{\bar{c}_0} = \frac{1}{2} e^{-\frac{Kx}{U}} \left[\operatorname{erf}\left(\frac{y + b/2}{2}\sqrt{\frac{U}{E_y x}}\right) - \operatorname{erf}\left(\frac{y - b/2}{2}\sqrt{\frac{U}{E_y x}}\right)\right] \tag{7-3}$$

Centerline concentration y = 0

$$\frac{\bar{c}_{max}}{\bar{c}_0} = e^{-\frac{Kx}{U}} \operatorname{erf}\left(\frac{b}{4}\sqrt{\frac{U}{E_y x}}\right) \tag{7-4}$$

Plume Width L(x) = $2\sqrt{3}\ \sigma_y(x)$

$$L/b = \sqrt{1 + \frac{24 E_y x}{U b^2}} \tag{7-5}$$

7.2.2 Non-homogenous turbulence $E_y = E_{y_0}\left(\frac{L}{b}\right)$

$$\frac{\bar{c}_{max}}{\bar{c}_0} = e^{-\frac{Kx}{U}} \operatorname{erf}\sqrt{\frac{3/2}{\left(1 + \frac{12 E_{y_0} x}{U b^2}\right)^2 - 1}} \tag{7-6}$$

$$L/b = 1 + \frac{12 E_{y_0} x}{U b^2} \tag{7-7}$$

7.2.3 Non-homogenous turbulence $E = E_{y_0}\left(\frac{L}{b}\right)^{4/3}$ "4/3 power law"

$$\frac{\bar{c}_{max}}{c_0} = e^{-\frac{Kx}{U}} \operatorname{erf}\sqrt{\frac{3/2}{\left(1 + \frac{8 E_{y_0} x}{U b^2}\right)^3 - 1}} \tag{7-8}$$

$$L/b = \left[1 + \frac{8 E_{y_0} x}{U b^2}\right]^{3/2} \tag{7-9}$$

8 Instantaneous Volume Source

One-dimensional

Instantaneous injection of mass M over $l_1 < x < l_2$ producing initial concentration c_i

$$c_i = \frac{M}{A(l_2 - l_1)}$$

where A = cross-sectional area.

$$\frac{c}{c_i} = \frac{e^{-Kt}}{2}\left\{\operatorname{erf}\left[\frac{(x-l_1)-Ut}{\sqrt{4E_x t}}\right] - \operatorname{erf}\left[\frac{(x-l_2)-Ut}{\sqrt{4E_x t}}\right]\right\} \quad (8\text{-}1)$$

Semi-infinite domain $-\infty < x < 0$

$$\frac{c}{c_i} = \frac{1}{2}\left[1 - \operatorname{erf}\frac{(x-Ut)}{\sqrt{4E_x t}}\right] \quad (8\text{-}2)$$